

# Blocking-inspired supersymmetric actions: a status report

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We provide a status report on the advances in blocking-inspired supersymmetric actions. This is done at the example of interacting supersymmetric quantum mechanics as well as the Wess-Zumino model. We investigate in particular the implications of a nontrivial realisation of translational symmetry on the lattice in this approach. We also discuss the locality of symmetry generators.

## I. INTRODUCTION

There has been an increasing interest in lattice simulations of supersymmetric theories as potential high energy extensions of the standard model. As any quantum field theory supersymmetry (SUSY) has to be regularized, and the space-time lattice is an obvious non-perturbative choice. On the lattice, however, SUSY is plagued by different conceptual and practical problems, ranging from the explicit breaking of supersymmetry by the boundary conditions to the violation of the Leibniz and chain rule of differentiation by lattice derivative operators [1]. For the current status of lattice SUSY see, e.g., [2] and references therein.

In Ref. [3] three of us suggested to break supersymmetry in a controlled way similarly to the implementation of chiral symmetry for Ginsparg-Wilson (GW) fermions [4]. This approach uses blocking as well-known from the renormalization group context. Rather than trying to evaluate the corresponding effective action, the concept of [3, 4] is to focus on the modified symmetry obeyed by it. This leaves us with a symmetry relation, or Ward-Takahashi identity, corresponding to a modified symmetry transformation. It ensures the full symmetry in the continuum limit, i. e. in the limit of removing the regulator. Other solutions than the one from blocking, which is typically unknown, should be possible for it.

The aim of this approach is to represent SUSY on the lattice in a similar way as a solution of such a symmetry relation (eq. (11) below). Its nontrivial right hand side should account for the breaking of the Leibniz rule by any lattice difference operator forbidding an unmodified SUSY [13]. One of the main obstacles in finding solutions is the obviously nonlinear character of this relation for truly interacting, i. e. higher than quadratic, systems. The appearance of polynomial interactions of degree higher than in the original theory or even non-polynomial interactions is familiar from effective actions. Their contribution vanishes in the continuum limit.

Another aspect that needs to be investigated in this approach is locality of the resulting action. Like the polynomial form, also locality might be lost in a generic effective

action, e. g. with a sharp momentum cut-off in the regulator. Nevertheless the locality in terms of the exponential suppression of the interaction with the distance is considered a basic requirement of a lattice theory.

It may help to review conventional lattice actions from this point of view. Non-abelian lattice gauge actions with local gauge invariance are well-established. Only in some special cases these actions are derived from an explicit solution of the blocking from the continuum. A solution to the GW relation for chiral symmetry is the overlap operator, which can be obtained from five-dimensional domain wall fermions. These solutions are derived according to the symmetry relation of the blocked action, but not as a solution of the blocking. Hence the symmetry relation is enough to ensure the symmetric continuum limit. Note that for chiral symmetry the gauge fields are spectators, such that this action should be viewed as quadratic in the fermion fields. Therefore, the derivation is simple compared to the generic case.

Solutions of the general symmetry relation have been worked out in [3] for free field theories and constant fields in supersymmetric quantum mechanics. Here, we further pursue this approach. We discuss further properties of the modified symmetry relation and its solutions. We give a status report on the advances made so far and an assessment of the remaining obstacles.

The present work is structured as follows. In Section II we recall the formalism of [3]. It is extended by an alternative way to derive solutions in the trivial non-interacting case. The supersymmetric quantum mechanics is considered as an simple application of this general setup. In Section III we discuss applications of the present formalism to the continuum theory in the presence of a controlled supersymmetry breaking. This includes in particular the two-dimensional Wess-Zumino model. In Section IV we discuss necessity of the breaking of translational invariance in the present setting. In Section V we summarise our findings.

## II. THE SET-UP

### A. Blocking and the symmetry relation

In this section we briefly review the symmetry relation for arbitrary linear symmetries that has been obtained via blocking in [3]. We consider some continuum theory with fields  $\varphi^i(x)$  and classical action  $S_{\text{cl}}[\varphi]$ , whose generating functional reads,

$$Z[J] = \frac{1}{\mathcal{N}} \int d\varphi e^{-S_{\text{cl}}[\varphi] + \int dx J^i(x) \varphi^i(x)}, \quad (1)$$

where the index  $i$  sums over internal structures, Lorentz indices and species of fields. In the following we consider theories with a linear symmetry, to wit

$$\varphi \rightarrow \varphi + \tilde{\delta}\varphi, \quad (\tilde{\delta}\varphi)^i(x) = \epsilon \int dy \tilde{M}^{ij}(x, y) \varphi^j(y), \quad (2)$$

where  $\tilde{M}$  may mix different field species, in the case of SUSY it does mix fermions and bosons.

A renormalization group step with the regulator  $\mathcal{R}$  leads to the Wilsonian effective action  $S[\phi]$ :

$$e^{-S[\phi]} := \mathcal{N}_{\mathcal{R}} \int d\varphi e^{-\mathcal{R}[\varphi, \phi] - S_{\text{cl}}[\varphi]}, \quad (3)$$

where in general  $\phi^i$  are the fields of the regularized effective action. We consider a specific quadratic regulator that is defined as

$$\mathcal{R}[\varphi, \phi] := \frac{1}{2} (\phi - \Phi[\varphi, f])_n^i \alpha_{nm}^{ij} (\phi - \Phi[\varphi, f])_m^j \quad (4)$$

$$\mathcal{N}_{\mathcal{R}} := \text{SDet}^{1/2} \alpha. \quad (5)$$

In our approach the blocked fields  $\phi_n^i$  are the fields on the lattice. Therefore, the regulator involves as a first step an averaging of the fields with an averaging function  $f$  around a lattice point  $an$ ,

$$\Phi[\varphi, f]_n^i := \int d^d x f^{ij}(an - x) \varphi^j(x), \quad (6)$$

where  $a$  is the lattice spacing and  $n$  are integers labelling the lattice sites.  $f$  may mix the different fields. The blocking kernel  $\alpha$  is assumed not to mix bosons and fermions.  $\text{SDet}$  is the super-determinant, i.e. the determinant for bosons and its inverse for fermions. When  $\alpha \rightarrow \infty$ , the lattice fields  $\phi$  are forced to be equal to the averaged fields  $\Phi$  (absorbing the superdeterminant) and eq. (3) becomes

$$e^{-S[\phi]} = \int d\varphi \prod_{i,n} \delta(\phi_n^i - \Phi[\varphi, f]_n^i) e^{-S_{\text{cl}}[\varphi]}. \quad (7)$$

When the lattice contains only one lattice site and the averaging function  $f$  is constant, the corresponding  $S$  is the constraint-effective potential [5].

At linear order a continuum symmetry transformation, eq. (2), on the r. h. s. of eq. (3) leads to the relation

$$e^{S[\phi]} \left\langle \int dx \tilde{\delta}\varphi^i(x) \frac{\delta \mathcal{R}[\varphi, \phi]}{\delta \varphi^i(x)} \right\rangle_{\mathcal{R}} - \langle \text{Tr} \tilde{M} \rangle_{\mathcal{R}} = 0, \quad (8)$$

where the expectation values at fixed blocked fields is defined as

$$\langle \mathcal{O} \rangle_{\mathcal{R}} := \mathcal{N}_{\mathcal{R}} \int d\varphi e^{-\mathcal{R}[\varphi, \phi] - S_{\text{cl}}[\varphi]} \mathcal{O}[\varphi; \phi]. \quad (9)$$

These expectation values include a functional dependence on the blocked fields  $\phi$ .

The continuum expectation values in relation (8) can be transferred into functional derivatives of the lattice fields only, provided the following lattice counterpart  $M$  of the continuum symmetry operator  $\tilde{M}$  can be defined,

$$M_{nm}^{ik} \Phi[\varphi, f]_m^k = \Phi[\tilde{M}\varphi, f]_n^i. \quad (10)$$

In [3] this has been named ‘additional constraint’ and its importance has been discussed for chiral symmetry and SUSY.

Then, as derived in [3], the relation (8) translates into

$$M_{nm}^{ij} \phi_m^j \frac{\delta S}{\delta \phi_n^i} = (M\alpha^{-1})_{nm}^{ij} \left( \frac{\delta S}{\delta \phi_m^j} \frac{\delta S}{\delta \phi_n^i} - \frac{\delta^2 S}{\delta \phi_m^j \delta \phi_n^i} \right) + \text{STr} M - \langle \text{STr} \tilde{M} \rangle_{\mathcal{R}}. \quad (11)$$

It will be interpreted as an invariance of the lattice action  $S$  and named ‘symmetry relation’ or ‘WT identity’ in due course.

All actions  $S$  defined via the blocking (3) will automatically fulfil the relation (11). The opposite does not apply since the symmetry relation is but one of the functional relations a blocked action satisfies.

Note that the l. h. s. of this equation multiplied by the symmetry parameter  $\epsilon$  is just the variation of  $S$  to leading order  $O(\epsilon)$

$$\delta S = \delta \phi_n^i \frac{\delta S}{\delta \phi_n^i} = \epsilon M_{nm}^{ij} \phi_m^j \frac{\delta S}{\delta \phi_n^i}. \quad (12)$$

Thus the r. h. s. of the symmetry relation (11) modifies the naive symmetry  $\delta S = 0$ . The first line of the r. h. s. of (11) is independent of the averaging function  $f$ , rather caused by the blocking kernel  $\alpha$ . To be precise, the nonsymmetric part of  $\alpha$  generates this term, it is absent for symmetric kernels [3] or delta-blocking  $\alpha^{-1} \rightarrow 0$ , cf. eq. (7).

Specialising to the case of chiral symmetry with fermions  $\psi, \bar{\psi}$  as fields, the Dirac operator  $\bar{\psi} D \psi$  as a quadratic action in them and symmetries  $M, \tilde{M} \propto \gamma_5$  trivially fulfilling the additional constraint, the first and second line of the symmetry relation (11) are nothing but the GW relation and the index theorem, respectively [3]. Solutions of this relation such as the overlap operator [6] represent chiral fermions on the lattice. Its locality has been shown for not too rough configurations [7].

Our hope is to represent SUSY on the lattice in a similar manner through actions that obey the symmetry relation (11) and approach the original (classical) action in the continuum limit.

The symmetry may be looked at in a slightly different way, namely by defining (‘deformed’) *field-dependent* symmetry transformations for the fields,

$$(M_{\text{def}})^{ij}_{nm} \phi_m^j = M_{nm}^{ij} \left( \phi_m^j - (\alpha^{-1})^{jk}_{mr} \frac{\delta S}{\delta \phi_r^k} \right), \quad (13)$$

which are generalisations of the modified chiral transformations on the lattice [8]. The WT identity (11) then reduces to

$$(M_{\text{def}})^{ij}_{nm} \phi_m^j \frac{\delta S}{\delta \phi_n^i} = (-1)^{|\phi^i||\phi^j|} \frac{\delta}{\delta \phi_n^i} [(M_{\text{def}})^{ij}_{nm} \phi_m^j] - \text{STr } \tilde{M}. \quad (14)$$

Note that for Wilson fermions – which are ultralocal but do not obey the naive chiral symmetry, as governed by the Nielsen-Ninomiya theorem [9, 10] – one can still write down a deformed symmetry, which, however, is non-local [3].

### B. A short note on trivial solutions

The relation eq. (11) leads to a deformed lattice symmetry that contains only derivatives with respect to the lattice fields. The only direct reference to the continuum is encoded in eq. (10). In order to find a different way to relate the lattice expression with the continuum we turn back to an intermediate step in the derivation of eq. (11).

The regulator (4) leads to the following relation of the expectation values

$$\left\langle \int dx \tilde{\delta} \varphi^i(x) \frac{\delta \mathcal{R}[\varphi, \phi]}{\delta \varphi^i(x)} \right\rangle_{\mathcal{R}} = \left\langle \Phi[\tilde{\delta} \varphi, f]_m^i \alpha_{nm}^{ij} (\Phi[\varphi, f] - \phi)_n^j \right\rangle_{\mathcal{R}} \quad (15)$$

or, equivalently,

$$\left\langle \int dx \tilde{\delta} \varphi^i \frac{\delta \mathcal{R}[\varphi, \phi]}{\delta \varphi^i(x)} \right\rangle_{\mathcal{R}} = (-1)^{|\phi^i||\tilde{\delta} \varphi^i|} \frac{\delta}{\delta \phi_m^i} \left\langle \Phi[\tilde{\delta} \varphi, f]_m^i \right\rangle_{\mathcal{R}}. \quad (16)$$

In some cases the functional dependence of the r. h. s. of eq. (16) on the lattice field  $\phi$  can be solved. In a theory without interactions the expectation value can be obtained from a saddle point approximation. The action has in this case the simple form

$$S_{\text{cl}} = \frac{1}{2} \int dx \varphi \tilde{K} \varphi, \quad (17)$$

and the expectation value from of the saddle point solution,  $\varphi_0$ , of the path integral leads to

$$\begin{aligned} \left\langle \Phi[\tilde{\delta} \varphi, f]_m^i \right\rangle_{\mathcal{R}} &= e^{-S} \epsilon f \tilde{M} \varphi_0 \\ &= e^{-S} \epsilon \left[ f \tilde{M} (f^T \alpha f + \tilde{K})^{-1} f \alpha \phi \right]. \end{aligned} \quad (18)$$

In this short hand notation the application of the averaging is represented as an application of  $f$  (i. e.  $f\varphi := \Phi[\varphi, f]$ ), cf. [3] for details. Hence  $\epsilon f \tilde{M} \varphi_0$  stands for the average of the variation of  $\varphi_0$ . With this simple solution the relation eq. (8) now becomes

$$\begin{aligned} & e^{S[\phi]} (-1)^{|\phi^i||\tilde{\delta} \varphi^i|} \frac{\delta}{\delta \phi_m^i} \left\langle \Phi[\tilde{\delta} \varphi, f]_m^i \right\rangle_{\mathcal{R}} - \epsilon \langle \text{STr } \tilde{M} \rangle_{\mathcal{R}} \\ &= \epsilon \left[ f \tilde{M} (f^T \alpha f + \tilde{K})^{-1} f \alpha \right]_{mn}^{ij} \phi_n^j \frac{\delta S}{\delta \phi_m^i} \\ &+ \epsilon \text{STr} \left[ f \tilde{M} (f^T \alpha f + \tilde{K})^{-1} f \alpha \right] - \epsilon \langle \text{STr } \tilde{M} \rangle_{\mathcal{R}} = 0 \end{aligned} \quad (19)$$

Since the relation is linear the matrix  $f \tilde{M} (f^T \alpha f + \tilde{K})^{-1} f \alpha$  can be interpreted as a symmetry generator on the lattice. Hence eq. (19) represents a symmetry relation on the lattice. The main difference to the modified symmetry relation (13) is that  $\tilde{K}$  encodes a direct reference to the continuum action. Such a reference seems to be unavoidable in a solution of the expectation values in (8). In the generic case these expectation values can not even be solved and only be approximations of them are available. In our approach the constraint in (10) allows to avoid the reference to the continuum action and to express the expectation values in terms of derivatives of lattice fields. In turn, keeping the reference to the continuum action, the additional constraint can be evaded.

Note that in a theory without interactions the lattice action defined in (3) follows directly by solving the Gaußian path integral. It is the perfect action as mentioned in [11]. This action is a solution of the lattice symmetry relations eq. (19) and eq. (13). Apart from the trace part the symmetry relation eq. (19) of the perfect action has already been found in [12].

### C. Supersymmetric Quantum Mechanics in the Continuum

Supersymmetric Quantum Mechanics (SUSYQM) is one of the simplest supersymmetric models and thus ideal for investigating SUSY on the lattice. It is a one-dimensional theory of a real boson  $\tilde{\chi}$ , a bosonic auxiliary field  $\tilde{F}$ , a complex fermion (Grassmannian)  $\tilde{\psi}$  and its complex conjugate  $\tilde{\bar{\psi}}$ , which we collect into the field vector  $\varphi^i = (\tilde{\chi}, \tilde{F}, \tilde{\psi}, \tilde{\bar{\psi}})$ . The off-shell action

$$\begin{aligned} S_{\text{cl}}[\varphi] &= \int dx \mathcal{L}_{\text{cl}}(\varphi(x)) \\ \mathcal{L}_{\text{cl}} &= \frac{1}{2} (\partial_x \tilde{\chi})^2 + \tilde{\bar{\psi}} \partial_x \tilde{\psi} - \frac{1}{2} \tilde{F}^2 + \tilde{\bar{\psi}} \frac{\partial W}{\partial \tilde{\chi}} \tilde{\psi} - \tilde{F} W, \end{aligned} \quad (20)$$

consists of kinetic terms (actually algebraic for the auxiliary field) and particular potential terms defined by the superpotential  $W(\tilde{\chi})$ . The latter encodes mass terms and interactions, for instance through the choice  $W(\tilde{\chi}) = m \tilde{\chi}^2 + g \tilde{\chi}^3$ .

This action is invariant under continuum supersymmetry transformations  $\delta\tilde{\chi} = -\bar{\epsilon}\tilde{\psi} + \epsilon\tilde{\bar{\psi}}$  etc. up to a surface term, which we collect into

$$\tilde{M} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\partial_x \\ -\partial_x & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\bar{M}} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -\partial_x & 0 \\ 0 & 0 & 0 & 0 \\ \partial_x & -1 & 0 & 0 \end{pmatrix}, \quad (21)$$

and obviously  $\{\tilde{M}, \tilde{\bar{M}}\} = 2\partial_x$ .

Let us parametrise the blocking first through its inverse as

$$a(\alpha^{-1})_{mn}^{ij} = \begin{pmatrix} a_2 & 0 & 0 & 0 \\ 0 & a_0 & 0 & 0 \\ 0 & 0 & 0 & a_1 \\ 0 & 0 & -a_1 & 0 \end{pmatrix}_{mn}. \quad (22)$$

The index of the parameters has been chosen according to the length dimension. This kernel can be shown to be the most general one up to  $\alpha$ 's that do not contribute to the symmetry relation, cf. App. F of [3]. The original  $\alpha$  is obviously

$$\frac{1}{a}\alpha_{mn}^{ij} = \begin{pmatrix} b_2 & 0 & 0 & 0 \\ 0 & b_0 & 0 & 0 \\ 0 & 0 & 0 & -b_1 \\ 0 & 0 & b_1 & 0 \end{pmatrix}_{mn}, \quad (23)$$

where  $b_{0,1,2} = 1/a_{0,1,2}$ .

### III. BLOCKING IN THE CONTINUUM

Our formalism can also be used to obtain equivalent theories in the continuum, which we explore in this section. The general structure of these solutions helps to identify the structure of possible solutions on the lattice and identify trivial transformations, that can be neglected. An advantage of this approach is that we do not need to consider the additional constraint.

To that end we do not perform an averaging, but replace eq. (6) by  $\Phi = \varphi$  (formally  $f$  is the delta distribution). Treating  $an$  as a continuous variable, all formulae can be taken over with obvious modifications (integrals

instead of sums etc.). In particular,  $M = \tilde{M}$  and no additional constraint occurs. Hence

$$e^{-S[\phi]} = \text{SDet}^{1/2}\alpha \int d\varphi e^{-\int dxdy (\phi-\varphi)(x)\alpha(x,y)(\phi-\varphi)(y)/2} \times e^{-S_{\text{cl}}[\varphi]}. \quad (24)$$

The transformed action  $S$  fulfils a relation analogous to (11),

$$\begin{aligned} \int dxdy \tilde{M}^{ij}(x,y) \phi^j(y) \frac{\delta S}{\delta \phi^i(x)} &= \\ \int dxdy (\tilde{M}\alpha^{-1})^{ij}(x,y) \left[ \frac{\delta S}{\delta \phi^j(y)} \frac{\delta S}{\delta \phi^i(x)} - \frac{\delta^2 S}{\delta \phi^j(y)\delta \phi^i(x)} \right], \end{aligned} \quad (25)$$

or, equivalently, is invariant under field-dependent deformed symmetry transformations

$$\begin{aligned} \int dy (\tilde{M}_{\text{def}})^{ij}(x,y) \phi^i(y) \\ = \int dy \tilde{M}^{ij}(x,y) \left[ \phi^j(y) - \int dz (\alpha^{-1})^{jk}(y,z) \frac{\delta S}{\delta \phi^k(z)} \right]. \end{aligned} \quad (26)$$

that can be written like (14),

$$\begin{aligned} \int dxdy (\tilde{M}_{\text{def}})^{ij}(x,y) \phi^j(y) \frac{\delta S}{\delta \phi^i(x)} &= -\text{STr} \tilde{M} \\ &+ (-1)^{|\phi^i||\phi^j|} \int dxdy \frac{\delta}{\delta \phi^i(x)} \left[ (\tilde{M}_{\text{def}})^{ij}(x,y) \phi^j(y) \right]. \end{aligned} \quad (27)$$

Even though the transformation does, strictly speaking, not correspond to a blocking of the degrees of freedom to lattice fields we still use the name blocking in due course to mark its similarity to the blocking transformation.

#### A. Blocking for SUSYQM

For SUSYQM the blocking (24) can be performed explicitly in the auxiliary and fermionic field, since the Lagrangian (20) is bilinear in this sector. For finite  $a_0$  and  $a_1$  (equivalently finite  $b_0$  and  $b_1$ ) proportional to  $\delta(x-y)$  and vanishing  $a_2$ /diverging  $b_2$  in the sense of (7) we obtain

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$$e^{-S[\chi, F, \psi, \bar{\psi}]} = \mathcal{N} \int d\tilde{F} d\tilde{\psi} d\tilde{\bar{\psi}} \exp \left( - \int dx \left[ \frac{b_0}{2} (F - \tilde{F})^2 + b_1 (\bar{\psi} - \tilde{\bar{\psi}})(\psi - \tilde{\psi}) + \mathcal{L}_{\text{cl}}(\chi, \tilde{F}, \tilde{\psi}, \tilde{\bar{\psi}}) \right] \right) \quad (28)$$

and thus

$$\begin{aligned} S &= \int dx \left\{ \frac{1}{2} (\partial_x \chi)^2 - \frac{1}{2} \frac{b_0}{b_0 - 1} F^2 - \frac{b_0}{b_0 - 1} F W - \frac{1}{2(b_0 - 1)} W^2 + \bar{\psi} \left[ b_1 - b_1^2 \left( \partial_x + \frac{\partial W}{\partial \chi} + b_1 \right)^{-1} \right] \psi \right\} \\ &- \log \det \left[ \partial_x + \frac{\partial W}{\partial \chi} + b_1 \right] \end{aligned} \quad (29)$$

This action has several interesting properties: first of all it depends on parameters  $b_{0,1}$  which in the limit  $b_{0,1} \rightarrow \infty$  – that is diverging  $\alpha$  – lead back to the original off-shell action. On the other hand, in the limit  $b_{0,1} \rightarrow 0$ , it is the on-shell action (with the fermionic action written as determinant), the auxiliary and fermionic field are just integrated out from the original action in (28).

The action fulfils the continuum symmetry relation (25) with nonvanishing right hand side. It is invariant under the field dependent transformation derived from (26),

$$\delta\chi = -\bar{\epsilon} \left( \psi + a_1 \frac{\delta S}{\delta\psi} \right) + \epsilon \left( \bar{\psi} - a_1 \frac{\delta S}{\delta\bar{\psi}} \right), \quad (30)$$

$$\delta F = -\bar{\epsilon} \partial_x \left( \psi + a_1 \frac{\delta S}{\delta\psi} \right) - \epsilon \partial_x \left( \bar{\psi} - a_1 \frac{\delta S}{\delta\bar{\psi}} \right), \quad (31)$$

$$\delta\psi = -\epsilon \partial_x \chi - \epsilon \left( F - a_0 \frac{\delta S}{\delta F} \right), \quad (32)$$

$$\delta\bar{\psi} = \bar{\epsilon} \partial_x \chi - \bar{\epsilon} \left( F - a_0 \frac{\delta S}{\delta F} \right). \quad (33)$$

Let us get some more intuition about this action by comparison with the nontrivial solution in the zero mode sector worked out in [3]. When reduced to constant fields, the action (29) on a space of volume  $V$  becomes

$$\begin{aligned} \frac{S}{V} = & -\frac{1}{b_0 - 1} \left[ \frac{b_0}{2} F^2 + b_0 F W + \frac{1}{2} W^2 \right] \\ & + \bar{\psi} \left( b_1 - \frac{b_1^2}{\frac{\partial W}{\partial \chi} + b_1} \right) \psi - \log \left( \frac{\partial W}{\partial \chi} + b_1 \right). \end{aligned} \quad (34)$$

To specialize to a term  $\lambda \bar{\psi} \chi \psi$  as in eq. (118) of [3], we choose

$$\frac{\partial W}{\partial \chi} + b_1 = \frac{b_1^2}{b_1 - \lambda \chi}. \quad (35)$$

A particular solution is

$$W = -\frac{b_1^2}{\lambda} \log \left( 1 - \frac{\lambda \chi}{b_1} \right) - b_1 \chi. \quad (36)$$

Plugged into the action this yields

$$\begin{aligned} S = & -\frac{1}{b_0 - 1} \left[ \frac{b_0}{2} F^2 - b_0 F \left( \frac{b_1^2}{\lambda} \log \left( 1 - \frac{\lambda \chi}{b_1} \right) + b_1 \chi \right) \right. \\ & \left. + \frac{1}{2} \left( \frac{b_1^2}{\lambda} \log \left( 1 - \frac{\lambda \chi}{b_1} \right) + b_1 \chi \right)^2 \right] \\ & - \log \left( \frac{b_1^2}{b_1 - \lambda \chi} \right) + \lambda \bar{\psi} \chi \psi. \end{aligned} \quad (37)$$

which is very similar to the bosonic part of the interacting solution in the zero mode sector given in Eq. (116) of [3],

$$\begin{aligned} h(\chi, F) = & \frac{1}{2} F^2 - \frac{1 + a_0}{a_1} \chi F + \frac{a_0(1 + a_0)}{2a_1^2} \chi^2 \\ & - \left( 1 + \frac{1 + a_0}{a_1^2 \lambda} F - \frac{a_0(1 + a_0)}{a_1^3 \lambda} \chi \right) \log(1 - a_1 \lambda \chi) \\ & + \frac{a_0(1 + a_0)}{2a_1^4 \lambda^2} (\log(1 - a_1 \lambda \chi))^2, \end{aligned} \quad (38)$$

where we have set the number of lattice points and the lattice spacing to one ( $N = a = 1$ ).

Up to a constant, this bosonic part can be obtained from blocking by the choice

$$W = -\frac{b_1^2}{\lambda} \log \left( 1 - \frac{\lambda \chi}{b_1} \right) - b_1 \chi - \frac{1}{2(1 + b_0)} \tilde{F}, \quad (39)$$

which is necessarily  $\tilde{F}$ -dependent. Originally,  $W$  was assumed a function of the bosonic field only. In particular, the blocking (28) is not guaranteed to give the action (29) anymore. However,  $\tilde{F}$  is still Gaussian and integrating out results in the desired action. Note that an  $\tilde{F}$ -dependent  $W$  only yields a supersymmetric action for constant fields, because the variation of  $\tilde{F}$  vanishes in this case. This marks the difference between (39) and the solutions of the blocked actions derived from (20) where the  $\tilde{F}$ -dependent term does not appear.

The action (29) derived via blocking in the auxiliary and fermionic sector is nontrivial: It contains interactions of different kind than those in the original action (20), it depends on additional parameters  $b_{0,1}$  and obeys nontrivial symmetries. Yet it is fully equivalent to the original SUSYQM system (since it is connected to the latter by a Gaussian blocking). As such it may serve as an example, in which the same physical content including the symmetry is represented by an unconventional action and a nonlinear symmetry relation.

The solution for constant fields can also help to understand the general difference between actions fulfilling the symmetry relation and actions derived from blocking. The action (38) is a solution of the symmetry relation for constant fields. Actions obtained from the blocking, on the other hand, are of the form (34). From the different  $F^2$  coefficients one immediately infers that the action (38) cannot be obtained from the blocking.

For most of the numerical simulations, however, the differences between the actions (20) and (29) are negligible. Usually the auxiliary field is integrated out and fermions are replaced by the determinant of the operator in between them. When both computations are performed one gets up to constants,

$$S_{\text{on}} = \int dx \left[ \frac{1}{2} (\partial_x \chi)^2 + \frac{1}{2} W^2 \right] - \log \det \left[ \partial_x + \frac{\partial W}{\partial \chi} \right], \quad (40)$$

from both actions, (29) and (20).



Technically this comes about because the fields  $\tilde{F}$  are integrated out in the blocking. For the on-shell action the new fields, say  $F$ , are integrated out as well,

$$\begin{aligned} \int dF e^{-S[F]} &= \int dF d\tilde{F} e^{-b_0(F-\tilde{F})^2/2 - S_{\text{cl}}[\tilde{F}]} \\ &= \int dG d\tilde{F} e^{-b_0 G^2/2 - S_{\text{cl}}[\tilde{F}]} \propto \int d\tilde{F} e^{-S_{\text{cl}}[\tilde{F}]}, \end{aligned} \quad (41)$$

such that the resulting expression is proportional to the one obtained by integrating out exactly those fields from the original action. Consequently,  $b_0$  (and likewise the fermionic  $b_1$ ) parametrising the family of actions (29) disappeared.

The nontrivial effect of the blocking reappears in the expectation values of the transformed fields  $\psi$ ,  $\bar{\psi}$ , and  $F$ . Part of the fermionic contribution of the action has been converted to a nonlocal term of the bosonic part. The nonlinear symmetry (33) is reflected in terms of complicated Ward identities relating the expectation values of the transformed fields.

Discretising the action (40) on the lattice, one is still confronted with all the mentioned problems of lattice SUSY. We have, however, gained some information about the general solutions one might expect from a blocking transformation. A less trivial reformulation would be to block the boson field by virtue of a finite  $b_2$ . This, however, requires to compute a non-Gaussian path integral, which is as difficult as solving the system itself.

### B. Two dimensional Wess-Zumino model

Blockings of the auxiliary field can always be applied in a straightforward way. Consider for example the Wess-Zumino model in two dimensions with complex bosonic fields  $\chi$ ,  $\tilde{F}$  and complex two component spinors  $\Psi$ , see e.g. [14, 15]. The action can be written in the following way

$$\begin{aligned} S_{\text{cl}} &= \int d^2x \left[ -\frac{1}{2}\chi^* \partial^2 \chi - \frac{1}{2}|\tilde{F}|^2 + \frac{1}{2}\tilde{F}W' + \frac{1}{2}\tilde{F}^*(W')^* \right. \\ &\quad \left. + \bar{\Psi}(\not{\partial} + P_+W'' + P_-(W'')^*)\Psi \right], \end{aligned} \quad (42)$$

where  $P_{\pm} = \frac{1}{2}(1 \pm \gamma_*)$  and  $W(\chi)$  is some polynomial of the field  $\chi$ . The blocking can now be easily applied in the auxiliary field sector:

$$\exp\left(\frac{1}{2} \int dx (F^* - \tilde{F}^*) b_0 (F - \tilde{F})\right) \quad (43)$$

This leads to the following action

$$\begin{aligned} S_{\text{cl}} &= \int d^2x \left[ -\frac{1}{2}\chi^* \partial^2 \chi - \frac{b_0}{2(1+b_0)}|\tilde{F}|^2 + \frac{1}{2(1+b_0)}|W'|^2 \right. \\ &\quad \left. + \frac{b_0}{2(1+b_0)}(\tilde{F}W' + \tilde{F}^*(W')^*) \right. \\ &\quad \left. + \bar{\Psi}(\not{\partial} + P_+W'' + P_-(W'')^*)\Psi \right]. \end{aligned} \quad (44)$$

In the limit of  $b_0 \rightarrow 0$  this is the on-shell action; in the limit of  $b_0 \rightarrow \infty$  the off-shell action. The symmetry transformations can be deduced straightforwardly and contain the nonlinear terms like the on-shell supersymmetry transformations. This shows how our setup can be generalized to more than one dimension. Again the difference of the transformed (blocked) and the original action are only relevant for expectation values of the auxiliary field.

## IV. TRANSLATIONAL INVARIANCE BLOCKED

Supersymmetry is intimately connected to the Poincaré algebra and thus to infinitesimal translations. Typically two SUSY transformations anticommute to a partial derivative. In this section we consider the anticommutator of two general linear symmetry generators and show that a corresponding symmetry relation also holds for it. In that way one is lead from SUSY to the symmetry relation for translations. It is easier to analyse this relation, but we will discuss that it can be fulfilled only under non-standard circumstances.

Let  $\epsilon_I \tilde{M}_I$  and  $\epsilon_{II} \tilde{M}_{II}$  be two infinitesimal continuum symmetries in the sense of (2), which both fulfill the additional constraint (10). It follows straightforwardly that every polynomial of these symmetries fulfills this constraint, too. For SUSY with its Grassmannian  $\epsilon$ 's this means that the anticommutator of  $\tilde{M}_I$  and  $\tilde{M}_{II}$  fulfills the additional constraint.

Consider the effect of a combination of the two symmetries. The simplest way to derive the corresponding symmetry relation is to use the fact that in the original theory infinitesimal symmetry generators form an algebra of the associated symmetry group. It is a necessary, but not sufficient, condition for the symmetry relation that the anticommutator of two symmetry transformation also fulfills a corresponding symmetry relation. Therefore, the commutator of these generators is again a generator and as such it is subject to a symmetry relation. For supersymmetry we write

$$[\epsilon_I \tilde{M}_I, \epsilon_{II} \tilde{M}_{II}] = \epsilon_I \epsilon_{II} \left\{ \tilde{M}_I, \tilde{M}_{II} \right\} \equiv \epsilon_I \epsilon_{II} \partial \quad (45)$$

defining the continuum operator  $\partial$ . In SUSYQM we have

$$\tilde{M}_I = \tilde{M}, \quad \tilde{M}_{II} = \tilde{\bar{M}}, \quad \partial = 2\partial_x. \quad (46)$$

For all supersymmetric theories one can find  $\tilde{M}$ 's that anticommute to partial derivatives wrt. coordinates. Therefore, we are left to analyse the consequences of the symmetry relation for partial derivatives. Obviously this can be done with just one field species.

In the following we investigate what kind of nontrivial modifications of the translational invariance on the lattice can be deduced from the symmetry relation. In particular we allow for a nonlocal antihermitian part of the operator  $\nabla$ . Such a nonlocal part arises in a natural way from the additional constraint. This consequence of the constraint for derivative operators have already been studied in [3].

Let us restrict our investigation to one real bosonic field  $\phi$  with values  $\phi_n$  at lattice sites  $an$ . Then the symmetry relation reads,

$$\nabla_{nm}\phi_m\frac{\delta S}{\delta\phi_n} = (\nabla\alpha^{-1})_{nm}\left(\frac{\delta S}{\delta\phi_m}\frac{\delta S}{\delta\phi_n} - \frac{\delta^2 S}{\delta\phi_m\delta\phi_n}\right) + \text{Tr}\nabla - \text{Tr}\partial. \quad (47)$$

As the second line of this equation is field-independent, we can focus on the first line.

Difference operators  $\nabla$  are assumed to be antihermitian (giving purely imaginary eigenvalues like partial derivatives do) and real, hence

$$\nabla^T = -\nabla, \quad \nabla_{nm} = -\nabla_{mn} \quad (48)$$

By construction,  $\alpha$  is symmetric and so is its inverse

$$(\alpha^{-1})^T = \alpha^{-1}, \quad (\alpha^{-1})_{nm} = (\alpha^{-1})_{mn}, \quad (49)$$

For the product on the r.h.s. of the symmetry relation (47) it follows that

$$(\nabla\alpha^{-1})^T = -\alpha^{-1}\nabla. \quad (50)$$

Translational invariant lattice operators  $X$  are circulant matrices,  $X_{m,n} = X_{m-n}$ , that anticommute with the matrix

$$P = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}. \quad (51)$$

Consequently, all powers of  $X$  including its inverse commute with  $P$ .

If we assume that both  $\nabla$  as well as  $\alpha$  and hence  $\alpha^{-1}$  are circulant,

$$[\nabla, P] = 0, \quad \nabla_{n,m} = \nabla_{n-m} \quad (52)$$

$$[\alpha, P] = 0, \quad [\alpha^{-1}, P] = 0, \quad (\alpha^{-1})_{nm} = (\alpha^{-1})_{n-m} \quad (53)$$

then  $\nabla$  and  $\alpha^{-1}$  commute and (50) turns into the anti-symmetry

$$(\nabla\alpha^{-1})^T = -\nabla\alpha^{-1}, \quad (\nabla\alpha^{-1})_{nm} = -(\nabla\alpha^{-1})_{mn}. \quad (54)$$

Since the first and second derivative of the lattice action  $S$  appearing on the r.h.s. of (47) are symmetric in  $n$  and  $m$ , it follows that the first line on the r.h.s. of the symmetry relation vanishes. The same conclusion can be drawn for fermion fields, with the obvious assumptions that  $\alpha$  and  $S$  connecting  $\bar{\psi}$  and  $\psi$  are antisymmetric.

In this case one is back at the naive symmetry relation, where the variation of the lattice action under infinitesimal translations must vanish. This immediately leads to nonlocal actions if  $\nabla$  is nonlocal.

There are several conceivable ways out keeping a non-trivial r.h.s. of the symmetry relation:

1.  $\alpha^{-1}$  is chosen non-circulant;
2.  $\nabla$  is non-circulant;
3. both  $\alpha^{-1}$  and  $\nabla$  are non-circulant;
4.  $\nabla$  is given a hermitian part.

All of them are not very natural. The first three mean that the difference operator in the lattice SUSY transformations or/and the blocking are not translational invariant.

For option 1 we can even show that the action cannot be translational invariant anymore. For that purpose we move to momentum space, where the symmetry relation reads

$$\nabla_p\phi_p\frac{\partial S}{\partial\phi_p} = \nabla_p(\alpha^{-1})_{pq}\left(\frac{\delta S}{\delta\phi_p}\frac{\delta S}{\delta\phi_{-q}} - \frac{\delta^2 S}{\delta\phi_p\delta\phi_{-q}}\right), \quad (55)$$

(no sum in  $p$ ) neglecting the trace parts. If the action  $S$  is translational invariant, the l.h.s. only contains field products whose momenta sum up to zero. This is not true for the r.h.s., because  $\alpha^{-1}$  connects different momenta. Thus the two sides can only be equal for all field values if the action is not translational invariant.

Option 4 resembles the Wilson-Dirac operator in gauge theories, where a hermitian part is added to the naive one in order to lift doublers with the disadvantages of explicitly breaking chiral symmetry and mixed hermiticity with complex eigenvalues.

The option 4 is beyond the scope of our current setup. In the current setup the lattice derivative operator  $\nabla$  is completely fixed by the additional constraint, cf. [3] for details. A hermitian part of the operator can not be added in this case. Hence it immediately follows from the above discussion that a local lattice version of the symmetry relation is not possible. However, the above discussion points to possible modifications of our approach. A systematic modification might be possible in the context of Section II B or as an approximate solution of the additional constraint. Such an approximation can be compared with the truncations of the nonlocal solutions of the naive symmetry relation as derived in [16].

## V. CONCLUSIONS

We briefly summarise our findings and discuss possible extensions of the present work. We have implemented the modified supersymmetry relations derived in [3] within low-dimensional interacting supersymmetric theories. We have also presented an alternative derivation of the lattice symmetry. Its disadvantage is a direct dependence on the underlying continuum theory. Hence it is not a genuine lattice symmetry. On the other hand it evades the additional constraint introduced in [3].

Staying in the continuum, i.e. without the averaging, we have studied SUSY systems that are equivalent through quadratic blockings. We have seen that nontrivial solutions emerge already in this simple case.

In the zero mode sector we have compared such blocked actions to solutions of the deformed symmetry relation from [3]. We emphasise that the symmetry relation constitutes only one of infinitely many functional relations between correlation functions, whereas the blocking in the path integral determines all of them. This entails that there are more solutions to the symmetry relation than actions obtained directly through blockings, an example of this has been given in the zero mode sector. However, solutions of the symmetry relation imply a supersymmetry-improved lattice action which can be the starting point for simulations.

The goal of our investigations has been to find local solutions of the deformed symmetry lattice SUSY. In [3] we have shown that the modified symmetry relations are satisfied for actions with algebraically decaying kinetic operators; hence in a strict sense locality is broken in these cases. It is again illustrative to compare this with the situation for chiral fermions. There, the locality of the Dirac operator is tightly linked to the locality of the generator of the deformed symmetry. For example, one can also derive a deformed symmetry relation for Wilson fermions but the related generator of the deformed

symmetry is not local. In turn, the generator of chiral symmetry for Ginsparg-Wilson fermions is local.

In the case of deformed supersymmetry the missing locality reflects the problem with the Leibniz rule and the fact that supersymmetry transformations do not form a compact group, as they include translations. We have investigated the symmetry relation for translations and elaborated that the deformed symmetry for them is the trivial one unless unconventional difference operators are used. In any case the nonlocality of the relation can not be avoided.

This immediately raises the question for the proper definition of the necessary locality or decay properties. In our opinion this is one key issue for practical implementation of lattice supersymmetry by means of blocked symmetry relations. Another way out is to derive symmetry relations that depend on the specific continuum action of the model under consideration. As stated in Section II B this requires the solution of specific expectation values. For nontrivial theories only approximations of these can be derived and it remains questionable if the resulting symmetry relation is enough to ensure the complete continuum symmetry.

In summary we have pushed forward the blocking-inspired approach towards lattice supersymmetry. In our opinion the classification of smooth (and hard) breakings of supersymmetry from the properties of the generator of the deformed supersymmetry would pave a way towards a practical implementation of lattice supersymmetry. In this sense we have advanced towards posing the right question but have not achieved an answer yet.

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- [1] P. H. Dondi and H. Nicolai, *Nuovo Cim.* **A41** (1977) 1.
  - [2] S. Catterall, *PoS LATTICE* **2010**, 002 (2010) [arXiv:1010.6224 [hep-lat]].
  - [3] G. Bergner, F. Bruckmann and J. M. Pawłowski, *Phys. Rev. D* **79** (2009) 115007 [arXiv:0807.1110 [hep-lat]].
  - [4] P. H. Ginsparg, K. G. Wilson, *Phys. Rev.* **D25**, 2649 (1982).
  - [5] L. O’Raifeartaigh, A. Wipf and H. Yoneyama, *Nucl. Phys. B* **271**, 653 (1986).
  - [6] H. Neuberger, *Phys. Lett. B* **417**, 141 (1998) [hep-lat/9707022], H. Neuberger, *Phys. Lett. B* **427**, 353 (1998) [hep-lat/9801031].
  - [7] P. Hernandez, K. Jansen and M. Lüscher, *Nucl. Phys.* **B552** (1999) 363; H. Neuberger, *Phys. Rev.* **D61** (2000) 085015; D. H. Adams, *Phys. Rev.* **D68** (2003) 065009.
  - [8] M. Lüscher, *Phys. Lett. B* **428**, 342 (1998) [hep-lat/9802011].
  - [9] H. B. Nielsen and M. Ninomiya, *Phys. Lett.* **B105** (1981) 219; D. Friedan, *Commun. Math. Phys.* **85** (1982) 481-490.
  - [10] O. Jahn and J. M. Pawłowski, *Nucl. Phys.* **B642** (2002) 357.
  - [11] W. Bietenholz, R. Brower, S. Chandrasekharan and U. J. Wiese, *Nucl. Phys. Proc. Suppl.* **53** (1997) 921 [arXiv:hep-lat/9608068].
  - [12] P. Hasenfratz, F. Niedermayer and R. von Allmen, *JHEP* **0610** (2006) 010 [hep-lat/0606021].
  - [13] G. Bergner, *JHEP* **1001** (2010) 024 [0909.4791 [hep-lat]].
  - [14] G. Bergner, T. Kaestner, S. Uhlmann and A. Wipf, *Annals Phys.* **323** (2008) 946 [0705.2212 [hep-lat]].
  - [15] S. Catterall and S. Karamov, *Phys. Rev. D* **68** (2003) 014503 [hep-lat/0305002].
  - [16] S. Schierenberg and F. Bruckmann, arXiv:1210.5404 [hep-lat].